

# ON SELF-DUAL SIMPLE TYPES OF $p$ -ADIC CLASSICAL GROUPS

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**ABSTRACT.** Let  $G$  be a classical group over a non-Archimedean local field of odd residual characteristic. Using recent work of S. Stevens, we define a certain kind of semisimple stratum, called *good*, and show that it provides a simple type in  $G$  which is an analogue of the simple type for  $GL(N, F)$  defined by Bushnell and Kutzko. Furthermore, we define a *self-dual* simple type in  $G$ .

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## INTRODUCTION

In order to classify the smooth representations of the general linear group  $G = GL(N, F)$  of a non-Archimedean local field  $F$ , a family of representations of certain compact open subgroups of  $G$ , which are called *simple types*, were constructed by Bushnell and Kutzko [1, 3]. Each simple type is associated with the inertial class  $[M, \sigma]_G$  of a supercuspidal representation  $\sigma$  of a Levi subgroup  $M$  in such a way that  $M \simeq GL(m, F) \times GL(m, F) \times \cdots \times GL(m, F)$  and  $\sigma \simeq \sigma_0 \times \sigma_0 \times \cdots \times \sigma_0$  for an irreducible supercuspidal representation  $\sigma_0$  of  $GL(m, F)$  with  $N = md$ .

For a symplectic group  $Sp(2N, F)$  over a non-Archimedean local field  $F$  of odd residual characteristic, a type is constructed in Blondel [4]. It is associated with the inertial class  $[M, \pi_0 \otimes \pi_0]$ , which consists of a maximal Levi subgroup  $M \simeq GL(N, F) \times GL(N, F)$  and the tensor product  $\pi_0 \otimes \pi_0$  of a self-contragredient irreducible supercuspidal representation  $\pi_0$  of  $GL(N, F)$ . These results are generalized to a maximal Levi subgroup of the other classical groups in Goldberg, Kutzko and Stevens [9], and to a Levi subgroup, which is not always maximal, of  $Sp(2N, F)$  in Blondel [5]. By the methods of Bushnell and Kutzko [1], in Kariyama [10], they can also be generalized to a Levi subgroup of an unramified unitary group over a non-Archimedean local field  $F_0$  of odd residual characteristic. Furthermore, in [14], many types in the general classical groups over  $F_0$  are exhibited.

The purpose of this article is to define a *good* skew semisimple stratum and construct a *simple type* attached to this stratum for the general classical groups  $G$  over  $F_0$  by using the results of [14]. This simple type plays the same role as that for  $GL(N, F)$ . Furthermore, following a result in [10], a *self-dual* simple type in  $G$  can be defined. This generalizes those types constructed in [4, 5, 9, 10].

Let  $(V, h)$  be the non-degenerate Hermitian form that defines the group  $G$ . We take a skew semisimple stratum  $[\Lambda, n, 0, \beta]$ , which consists of a lattice sequence  $\Lambda$  in  $V$ , integers  $n > 0$ , and a semisimple skew element  $\beta = \sum_{i=1}^{\ell+1} \beta_i$  in  $\text{End}_F(V)$ . The  $F$ -algebra  $F[\beta]$  generated by  $\beta$  decomposes  $V$  into a direct sum of simple modules  $V = \bigoplus_{i=1}^{\ell+1} V^i$ . We say that  $[\Lambda, n, 0, \beta]$  is *good*, if  $V$  has another decomposition  $V = \bigoplus_{j=-m}^m W^{(j)}$  which is *self-dual*, *exactly subordinate* to  $[\Lambda, n, 0, \beta]$  in the sense

of [14] and has the property that for  $j \neq 0$ ,  $W^{(j)}$  is contained in  $V^{\ell+1}$  and has constant  $F$ -dimension (see Definition 2.1).

The latter decomposition of  $V$  yields a parabolic  $P = MU$  of  $G$  with  $M \simeq G^{(0)} \times GL(N/m, F)^m$ , for some positive integers  $m, N$  with  $m|N$ , where  $G^{(0)}$  is a subgroup of the same type as  $G$ . From the lattice  $\Lambda$ , we obtain compact open subgroups  $J_P^1 \subset J_P$  of  $G$  such that  $J_P/J_P^1 \simeq \overline{G}^{(0)} \times GL(f, k_{E'})^m$ , for some positive integer  $f$  and a finite field  $k_{E'}$ , and a certain irreducible representation  $\kappa_P$  of  $J_P$ , as in [14].

For an irreducible representation  $\tau$  of  $J_P$  that is trivial on  $J_P^1$ , we say that the representation  $\lambda_P = \kappa_P \otimes \tau$  of  $J_P$  is a *simple type* if  $\tau$  induces a representation of  $J_P/J_P^1$  that contains a certain irreducible cuspidal representation of the identity component of  $J_P/J_P^1$  as a finite reductive group (Definition 4.5). If  $(J_P, \lambda_P)$  is a simple type, there exist irreducible supercuspidal representations  $\pi_{\text{cusp}}, \tilde{\pi}^{(1)}, \dots, \tilde{\pi}^{(m)}$  of  $G^{(0)}, GL(N/m, F)$ , respectively, such that  $(J_P, \lambda_P)$  is a  $[M, \pi_M]_G$ -type in  $G$  in the sense of Bushnell-Kutzko [2], where  $\pi_M = \pi_{\text{cusp}} \otimes \bigotimes_{j=1}^m \tilde{\pi}^{(j)}$  (Theorem 6.3).

There exist certain Weyl group elements  $s_j$ , for  $1 \leq j \leq m$ , and the conjugation on  $M$  by  $s_j$  induces an involution  $\sigma_j$  on the  $j$ -th factor  $GL(N/m, F)$  of  $M$ . It leaves the  $j$ -th factor  $GL(f, k_{E'})$  of  $J_P/J_P^1$  stable. A simple type  $(J_P, \lambda_P)$ , with  $\lambda_P = \kappa_P \otimes \tau$ , is called *self-dual* if the  $j$ -th factor  $\tilde{\tau}^{(j)}$  on  $GL(f, k_{E'})$  of the representation  $\tau$  satisfies  $\tilde{\tau}^{(j)} \circ \sigma_j \simeq \tilde{\tau}^{(j)}$ , for  $1 \leq j \leq m$  (this is equivalent to Definition 5.2). We show that if  $\pi$  is an irreducible smooth representation of  $G$  that contains a self-dual simple type  $\lambda_P$ , then there exists an irreducible self-dual supercuspidal representation  $\rho$  of  $GL(N/m, F)$  and a representation  $\pi_{\text{cusp}}$  of  $G^{(0)}$  as above such that  $\pi$  is equivalent to a  $G$ -subquotient of the parabolically induced representation  $\text{Ind}_P^G(\pi_{\text{cusp}} \otimes \rho \nu^{x_1} \otimes \dots \otimes \rho \nu^{x_m})$ , where  $\nu = |\det|_F$  on  $GL(N/m, F)$  and  $x_1, \dots, x_m \in \mathbb{C}$ , (Theorem 6.2).

The remainder of this paper is structured as follows: In Section 1, we introduce notation and provide some required definitions. In Section 2, we define a good skew semisimple stratum. In Section 3, we recall the notion of a  $\beta$ -extension, which is defined in [14]. We define a simple type  $(J_P, \lambda_P)$  and a self-dual simple type in Sections 4 and 5. In Section 6, we prove that a simple type  $(J_P, \lambda_P)$  is a type in the sense of [2]. In fact it is a  $G$ -cover, as explained above.

*Notation:* We denote by  $\mathbb{N}$ ,  $\mathbb{Z}$ , and  $\mathbb{C}$  the set of natural numbers, the ring of rational integers, and the field of complex numbers, respectively. For a ring  $R$ , we denote the multiplicative group of  $R$  by  $R^\times$ .

## 1. PRELIMINARIES

We recall the notation used in Bushnell-Kutzko [1, 3], Stevens [13, 14].

Let  $F$  be a non-Archimedean local field with Galois involution  $\bar{\phantom{x}}$ , and with fixed field  $F_0 = \{x \in F \mid \bar{x} = x\}$ . We allow  $F_0$  to equal  $F$ . Let  $\mathfrak{o}_F$  be the ring of integers of  $F$ ,  $\mathfrak{p}_F$  the maximal ideal of  $\mathfrak{o}_F$ ,  $\varpi_F$  a uniformizer of  $F$ , and  $k_F = \mathfrak{o}_F/\mathfrak{p}_F$  the residue class field. We will assume throughout this study that the residual characteristic  $p$  of  $F$  is not equal to 2.

Let  $V$  be a finite-dimensional vector space over  $F$  equipped with a non-degenerate  $\varepsilon$ -hermitian form  $h$ , where  $\varepsilon \in \{+1, -1\}$ . Put  $A = \text{End}_F(V)$ , and let  $a \mapsto \bar{a}$  be the (anti-)involution on  $A$  defined by the form  $h$ :

$$h(av, w) = h(v, \bar{a}w) \quad (a \in A, v, w \in V).$$

Put  $\tilde{G} = A^\times = \text{Aut}_F(V)$ , and define an automorphism  $\sigma$  of order 2 on  $\tilde{G}$  by  $\sigma(g) = \bar{g}^{-1}$  ( $g \in \tilde{G}$ ). Put  $\Sigma = \{1, \sigma\}$ , where 1 denotes the identity on  $A$ , and set

$$G^+ = \tilde{G}^\Sigma = \{g \in \tilde{G} \mid \sigma(g) = g\}.$$

Denote by  $G$  the identity component of  $G^+$ . Then  $G$  is either a unitary group, a symplectic group, or a special orthogonal group over  $F_0$ . For a subgroup  $\tilde{H}$  of  $\tilde{G}$  with  $\sigma(\tilde{H}) = \tilde{H}$ , write  $H^+ = \tilde{H} \cap G^+$ ,  $H = \tilde{H} \cap G$ .

An  $\mathfrak{o}_F$ -lattice sequence in  $V$  is a function  $\Lambda : \mathbb{Z} \rightarrow \{\mathfrak{o}_F\text{-lattices in } V\}$  that satisfies

- (1)  $n \geq m$  implies  $\Lambda(n) \subset \Lambda(m)$ ,
- (2) there exists a positive integer  $e = e(\Lambda | \mathfrak{o}_F)$  (the  $\mathfrak{o}_F$ -period) such that  $\Lambda(n + e) = \mathfrak{p}_F \Lambda(n)$ , ( $n \in \mathbb{Z}$ ).

We say that an  $\mathfrak{o}_F$ -lattice sequence  $\Lambda$  is *strict*, if  $\Lambda(n) \supsetneq \Lambda(n+1)$  ( $n \in \mathbb{Z}$ ).

For an  $\mathfrak{o}_F$ -lattice  $L$  in  $V$ , we define the *dual lattice*  $L^\#$  by  $L^\# = \{v \in V \mid h(v, L) \subset \mathfrak{p}_F\}$ . An  $\mathfrak{o}_F$ -lattice sequence  $\Lambda$  in  $V$  is called *self-dual*, if there exists an integer  $d$  such that  $\Lambda(k)^\# = \Lambda(d - k)$  ( $k \in \mathbb{Z}$ ).

From an  $\mathfrak{o}_F$ -lattice sequence  $\Lambda$  in  $V$ , we obtain a filtration on  $A$  by

$$\mathfrak{a}_n = \mathfrak{a}_n(\Lambda) = \{x \in A \mid x\Lambda(k) \subset \Lambda(k+n) \text{ } (k \in \mathbb{Z})\} \text{ } (n \in \mathbb{Z}).$$

In particular,  $\mathfrak{a}_0$  is a hereditary  $\mathfrak{o}_F$ -order in  $A$  and  $\mathfrak{a}_1$  is its Jacobson radical. This filtration defines a valuation  $\nu_\Lambda$  on  $A$  by  $\nu_\Lambda(x) = \sup\{n \in \mathbb{Z} \mid x \in \mathfrak{a}_n\}$ , ( $x \in A$ ), with  $\nu_\Lambda(0) = +\infty$ .

From an  $\mathfrak{o}_F$ -lattice sequence  $\Lambda$  in  $V$ , we obtain open compact subgroups of  $\tilde{G}$  by

$$\begin{aligned} \tilde{P} &= \tilde{P}(\Lambda) = \mathfrak{a}_0(\Lambda)^\times, \\ \tilde{P}_n &= \tilde{P}_n(\Lambda) = 1 + \mathfrak{a}_n(\Lambda) \text{ } (n > 0). \end{aligned}$$

Then the  $\tilde{P}_n$  ( $n > 0$ ) are normal subgroups of  $\tilde{P}$  and form a filtration of  $\tilde{G}$ . If  $\Lambda$  is self-dual, then we obtain open compact subgroups of  $G^+$  and  $G$  from these groups by

$$\begin{aligned} P^+ &= P^+(\Lambda) = \tilde{P}(\Lambda) \cap G^+, \text{ } P = P(\Lambda) = \tilde{P}(\Lambda) \cap G, \\ P_n &= P_n(\Lambda) = \tilde{P}_n(\Lambda) \cap G \text{ } (n > 0). \end{aligned}$$

The quotient  $\mathcal{G} = P/P_1$  is the group of  $k_{F_0}$ -rational points of a reductive algebraic group defined over  $k_{F_0}$ , where  $k_{F_0}$  denotes the residue class field of  $F_0$ . We note that it is not always connected. We denote by  $P^\circ = P^\circ(\Lambda)$  the inverse image of the identity component  $\mathcal{G}^\circ$  of  $\mathcal{G} = P/P_1$  in  $P = P(\Lambda)$ . Then we have  $\mathcal{G}^\circ = P^\circ/P_1$ .

A *stratum* in  $A$  is a 4-tuple  $[\Lambda, n, r, b]$ , where  $\Lambda$  is an  $\mathfrak{o}_F$ -lattice sequence in  $V$ ,  $n \in \mathbb{Z}$ ,  $r \in \mathbb{Z}$  with  $n \geq r \geq 0$ , and  $b \in \mathfrak{a}_{-n}(\Lambda)$ . A stratum  $[\Lambda, n, r, \beta]$  is called *simple*, if it satisfies the following conditions:

- (1) the algebra  $E = F[\beta]$  is a field;
- (2)  $\Lambda$  is an  $\mathfrak{o}_E$ -lattice sequence (which we denote by  $\Lambda_{\mathfrak{o}_E}$ );
- (3)  $\nu_\Lambda(\beta) = -n$ ;
- (4)  $k_0(\beta, \Lambda) < -r$ ,

where  $k_0(\beta, \Lambda)$  is the integer defined in [13, §5] (cf. [1, (1.5)]).

A stratum  $[\Lambda, n, r, b]$  is called *null*, if  $n = r$  and  $b = 0$ .

Let  $[\Lambda, n, r, \beta]$  be a stratum in  $A$ , and  $V = \bigoplus_{i=1}^\ell V^i$  a direct sum decomposition of  $V$  into  $F$ -subspaces. We say that the  $F$ -decomposition  $V = \bigoplus_{i=1}^\ell V^i$  is *splitting* for  $[\Lambda, n, r, \beta]$ , if we have  $\Lambda(k) = \bigoplus_{i=1}^\ell \Lambda^i(k)$  ( $k \in \mathbb{Z}$ ),  $\beta = \sum_{i=1}^\ell \beta_i$ , where for each

$i$ ,  $\Lambda^i(k) = \Lambda(k) \cap V^i$  ( $k \in \mathbb{Z}$ ), and for the projection  $\mathbf{1}^i : V \rightarrow V^i$  with kernel  $\bigoplus_{j \neq i} V^j$ ,  $\beta_i = \mathbf{1}^i \beta \mathbf{1}^i$ .

**Definition 1.1.** ([13, 3.2]). A stratum  $[\Lambda, n, r, \beta]$  in  $A$  is called *semisimple*, if either it is null or  $\nu_\Lambda(\beta) = -n$  and there exists a splitting  $V = \bigoplus_{i=1}^\ell V^i$  for the stratum such that

- (1) for  $1 \leq i \leq \ell$ ,  $[\Lambda^i, q_i, r, \beta_i]$  is a simple or null stratum in  $A^i = \text{End}_F(V^i)$ , where  $q_i = r$  if  $\beta_i = 0$ ,  $\nu_{\Lambda^i}(\beta_i) = -q_i$  otherwise; and
- (2) for  $1 \leq i, j \leq \ell$ ,  $i \neq j$ , the stratum  $[\Lambda^i \oplus \Lambda^j, q, r, \beta_i + \beta_j]$  is not equivalent to a simple stratum or null stratum, with  $q = \max\{q_i, q_j\}$ .

A semisimple stratum  $[\Lambda, n, r, \beta]$  is called *skew*, if (1)  $\Lambda$  is self-dual, (2)  $\bar{\beta} = -\beta$ , and (3)  $V = \bigoplus_{i=1}^\ell V^i$  in Definition 1.1 is orthogonal with respect to the Hermitian form  $h$ .

Let  $[\Lambda, n, r, \beta]$  be a skew semisimple stratum in  $A$ . Denote by  $B$  the  $A$ -centralizer of  $\beta$ , and set  $\tilde{G}_E = B^\times \cap \tilde{G}$ ,  $G_E^+ = B^\times \cap G^+$ , and  $G_E = B^\times \cap G$ . Then we write

$$\begin{aligned} \tilde{P}(\Lambda_{\mathfrak{o}_E}) &= \tilde{P}(\Lambda) \cap \tilde{G}_E, \quad \tilde{P}_n(\Lambda_{\mathfrak{o}_E}) = \tilde{P}_n(\Lambda) \cap \tilde{G}_E \quad (n \geq 1), \\ P(\Lambda_{\mathfrak{o}_E}) &= \tilde{P}(\Lambda_{\mathfrak{o}_E}) \cap G_E, \quad P_n(\Lambda_{\mathfrak{o}_E}) = \tilde{P}_n(\Lambda_{\mathfrak{o}_E}) \cap G_E \quad (n \geq 1). \end{aligned}$$

Similarly,  $P^+(\Lambda_{\mathfrak{o}_E}) = \tilde{P}(\Lambda_{\mathfrak{o}_E}) \cap G_E^+$ . Denote by  $P^\circ(\Lambda_{\mathfrak{o}_E})$  the inverse image in  $P(\Lambda_{\mathfrak{o}_E})$  of the identity component of the quotient  $P(\Lambda_{\mathfrak{o}_E})/P_1(\Lambda_{\mathfrak{o}_E})$ .

## 2. GOOD SKEW SEMISIMPLE STRATA

We define a skew semisimple stratum in  $A$  which is called *good*. Assume that  $[\Lambda, n, 0, \beta]$  is a skew semisimple stratum in  $A$  with  $V = \bigoplus_{i=1}^{\ell+1} V^i$  and  $\beta = \sum_{i=1}^{\ell+1} \beta_i$  as splitting. Then, by definition, the Hermitian form  $h$  can be decomposed into an orthogonal direct sum:  $h = \bigoplus_{i=1}^{\ell+1} h_i$  on  $V = \bigoplus_{i=1}^{\ell+1} V^i$ , where each  $h_i$  is the restriction of  $h$  to  $V^i$ .

Let  $V = \bigoplus_{j=-m}^m W^{(j)}$  be a decomposition of  $V$  into subspaces such that

- (1)  $W^{(j)} = \bigoplus_{i=1}^{\ell+1} (W^{(j)} \cap V^i)$ , for  $-m \leq j \leq m$ , and  $V^i = \bigoplus_{j=-m}^m (W^{(j)} \cap V^i)$ , for  $1 \leq i \leq \ell + 1$ ,
- (2)  $W^{(j)} \cap V^i$  is an  $E_i$ -subspace of  $V^i$ , for  $-m \leq j \leq m$  and  $1 \leq i \leq \ell + 1$ .

By [14, 5.1], we say that  $V = \bigoplus_{j=-m}^m W^{(j)}$  is *properly subordinate to*  $[\Lambda, n, 0, \beta]$ , if the following conditions are satisfied:

- (1)  $\Lambda(n) = \bigoplus_{j=-m}^m \Lambda^{(j)}(n)$  ( $n \in \mathbb{Z}$ ), where  $\Lambda^{(j)}(n) = \Lambda(n) \cap W^{(j)}$ ,
- (2) For any integers  $k$  and  $i$ , with  $1 \leq i \leq \ell + 1$ , there exists at most one  $j$ ,  $-m \leq j \leq m$ , such that

$$(\Lambda(k) \cap W^{(j)} \cap V^i) \supsetneq (\Lambda(k+1) \cap W^{(j)} \cap V^i).$$

Moreover, by [14, 5.3],  $V = \bigoplus_{j=-m}^m W^{(j)}$  is called *self-dual*, if the orthogonal complement  $(W^{(j)})^\perp$  of  $W^{(j)}$  is equal to  $\bigoplus_{k \neq j} W^{(k)}$ , with respect to the form  $h$ .

**Definition 2.1.** Let  $[\Lambda, n, 0, \beta]$  be a skew semisimple stratum in  $A$  with  $V = \bigoplus_{i=1}^{\ell+1} V^i$  and  $\beta = \sum_{i=1}^{\ell+1} \beta_i$  a splitting. We say that the stratum  $[\Lambda, n, 0, \beta]$  is *good*, if there exists a self-dual decomposition  $V = \bigoplus_{j=-m}^m W^{(j)}$  which satisfies the following properties:

- (1)  $V = \bigoplus_{j=-m}^m W^{(j)}$  is *exactly subordinate* to  $[\Lambda, n, 0, \beta]$ , in the sense of [14, Definition 6.5], that is, it is minimal among all self-dual decompositions which are properly subordinate to  $[\Lambda, n, 0, \beta]$ ,
- (2) for  $j \neq 0$ ,  $W^{(j)}$  is contained in  $V^{\ell+1}$ ,
- (3) for  $j \neq 0$ ,  $\dim_{E_{\ell+1}} W^{(j)}$  are all the same, say  $f$ .

We assume that  $[\Lambda, n, 0, \beta]$  is a good skew semisimple stratum in  $A$ . Then by definition we have

$$(2.1) \quad W^{(0)} = \left( \bigoplus_{i=1}^{\ell} V^i \right) \oplus (W^{(0)} \cap V^{\ell+1}),$$

$$(2.2) \quad V^{\ell+1} = (W^{(0)} \cap V^{\ell+1}) \oplus \left( \bigoplus_{j=-m, j \neq 0}^m W^{(j)} \right),$$

where possibly  $W^{(0)} \cap V^{\ell+1} = (0)$ . Set  $E' = E_{\ell+1} = F[\beta_{\ell+1}]$ , and let  $N$  be the positive integer defined by

$$\dim_F \left( \bigoplus_{j=-m, j \neq 0}^m W^{(j)} \right) = 2N.$$

Since  $\dim_{E'}(W^{(j)}) = f$ , for  $j \neq 0$ , we have

$$(2.3) \quad N = m \dim_F(W^{(j)}) = m[E' : F]f,$$

where  $[E' : F]$  denotes the field extension degree of  $E'/F$ .

### 3. BETA EXTENSIONS

In this section, we assume that  $[\Lambda, n, 0, \beta]$  is a good skew semisimple stratum in  $A$  with  $V = \bigoplus_{i=1}^{\ell+1} V^i$  and  $\beta = \sum_{i=1}^{\ell+1} \beta_i$  a splitting, as defined in the previous section. Set  $E_i = F[\beta_i]$ , for  $1 \leq i \leq \ell+1$ , and  $E = \bigoplus_{i=1}^{\ell+1} E_i$ .

In [13], we have the  $\mathfrak{o}_F$ -lattices  $\tilde{\mathfrak{H}}^t = \tilde{\mathfrak{H}}^t(\beta, \Lambda)$ ,  $\tilde{\mathfrak{J}}^t = \tilde{\mathfrak{J}}^t(\beta, \Lambda)$  in  $A = \text{End}_F(V)$ , and the compact open subgroups  $\tilde{H}^t = \tilde{H}^t(\beta, \Lambda)$ ,  $\tilde{J}^t = \tilde{J}^t(\beta, \Lambda)$  of  $\tilde{G}$ , for  $t = 0, 1$ . Since these objects are  $\sigma$ -stable, we obtain compact open subgroups of  $G^+$  and  $G$  as follows:

$$J^+(\beta, \Lambda) = \tilde{J}(\beta, \Lambda) \cap G^+, \quad H^t(\beta, \Lambda) = \tilde{H}^t(\beta, \Lambda) \cap G, \quad J^t(\beta, \Lambda) = \tilde{J}^t(\beta, \Lambda) \cap G,$$

for  $t = 0, 1$ . Then  $J^+(\beta, \Lambda) = P^+(\Lambda_{\mathfrak{o}_E})J^1(\beta, \Lambda)$  and  $J(\beta, \Lambda) = P(\Lambda_{\mathfrak{o}_E})J^1(\beta, \Lambda)$  (cf. [14, 3.1]). Put  $J^0(\beta, \Lambda) = P^0(\Lambda_{\mathfrak{o}_E})J^1(\beta, \Lambda)$ . Then we have  $J^0(\beta, \Lambda) \subset J(\beta, \Lambda) \subset J^+(\beta, \Lambda)$ .

Denote by  $\tilde{\mathcal{C}}(\Lambda, 0, \beta)$  the set of all semisimple characters of  $\tilde{H}^1(\beta, \Lambda)$ , defined by [13, Definition 3.13]. Put

$$\mathcal{C}_-(\Lambda, 0, \beta) = \{\theta|_{H^1(\beta, \Lambda)} | \theta \in \tilde{\mathcal{C}}(\Lambda, 0, \beta) \text{ and } \theta^\sigma(x) = \theta(x) \ (x \in \tilde{H}^1(\beta, \Lambda))\},$$

where  $\theta^\sigma(x) = \theta(\sigma^{-1}(x))$ , which is defined as in [13, 3.6].

Let  $B$  be the  $A$ -centralizer of  $\beta = \sum_{i=1}^{\ell+1} \beta_i$ . Then we have  $B = \bigoplus_{i=1}^{\ell+1} B^i$ , where  $B^i = \text{End}_{E_i}(V^i)$ . As in [14, 4.2], we choose a self-dual  $\mathfrak{o}_F$ -lattice sequence  $\Lambda^M$  in  $V$  which satisfies (1)  $\mathfrak{b}_0(\Lambda^M) = \mathfrak{a}_0(\Lambda^M) \cap B$  is a maximal self-dual  $\mathfrak{o}_E$ -order of  $B = \bigoplus_{i=1}^{\ell+1} B^i$ , and (2)  $\mathfrak{b}_0(\Lambda^M) \supset \mathfrak{b}_0(\Lambda)$ . From [14, Corollary 2.9], there exists a self-dual  $\mathfrak{o}_E$ -lattice sequence  $\Lambda^m$  in  $V$  which satisfies (1)  $\mathfrak{b}_0(\Lambda^m)$  is a minimal self-dual  $\mathfrak{o}_E$ -order of  $B$ , and (2)  $\mathfrak{a}_0(\Lambda^m) \subset \mathfrak{a}_0(\Lambda)$ . Thus we have  $\mathfrak{b}_0(\Lambda^m) \subset \mathfrak{b}_0(\Lambda) \subset \mathfrak{b}_0(\Lambda^M)$ .

Since the invariants  $k_0(\beta, \Lambda^m)$  and  $k_0(\beta, \Lambda^M)$  are negative integers, there exist integers  $n_m, n_M$  such that  $[\Lambda^m, n_m, 0, \beta]$ ,  $[\Lambda^M, n_M, 0, \beta]$  are skew semisimple strata in  $A$ . Let  $\theta \in \mathcal{C}_-(\Lambda, 0, \beta)$ . Then from [13, 3.2], there exist canonical bijections  $\tau_{\Lambda, \Lambda^m, \beta} : \mathcal{C}_-(\Lambda, 0, \beta) \simeq \mathcal{C}_-(\Lambda^m, 0, \beta)$  and  $\tau_{\Lambda, \Lambda^M, \beta} : \mathcal{C}_-(\Lambda, 0, \beta) \simeq \mathcal{C}_-(\Lambda^M, 0, \beta)$ . Put  $\theta_m = \tau_{\Lambda, \Lambda^m, \beta}(\theta)$ ,  $\theta_M = \tau_{\Lambda, \Lambda^M, \beta}(\theta)$ .

**Proposition 3.1.** ([14, Proposition 3.5]) *There exists a unique irreducible representation  $\eta$  (resp.  $\eta_m, \eta_M$ ) of  $J^1 = J^1(\beta, \Lambda)$  (resp.  $J_m^1 = J^1(\beta, \Lambda^m)$ ,  $J_M^1 = J^1(\beta, \Lambda^M)$ ) containing  $\theta$  (resp.  $\theta_m, \theta_M$ ).*

From the above choice of  $\Lambda^m, \Lambda^M$ , we can form the group  $J_{m,M}^1 = P_1(\Lambda_{\mathfrak{o}_E}^m)J_M^1$ .

**Proposition 3.2.** ([14, Proposition 3.7 and Theorem 4.1]) (i) There exists a unique irreducible representation  $\eta_{m,M}$  of  $J_{m,M}^1$  satisfying (1)  $\eta_{m,M}|_{J_M^1} = \eta_M$ , (2)  $\eta_{m,M}$  and  $\eta_m$  induce equivalent irreducible representations of  $P_1(\Lambda^m)$ .

(ii) There exists a representation  $\kappa_M$  of  $J_M^+ = J^+(\beta, \Lambda^M)$  that extends  $\eta_{m,M}$ .

From [14, Definition 4.5], there exists an extension  $\kappa$  of  $\eta$  to  $J^+$ , which is called a  $\beta$ -extension, relative to  $\Lambda^M$ , and compatible with  $\kappa_M$ . The representation  $\kappa$  depends only on  $\Lambda^M$ , not on the choice of  $\Lambda^m$ .

#### 4. SIMPLE TYPES

Let  $[\Lambda, n, 0, \beta]$  be a good skew semisimple stratum in  $A$  with the splitting  $V = \bigoplus_{i=1}^{\ell+1} V^i$  and  $\beta = \sum_{i=1}^{\ell+1} \beta_i$ , defined in section 2. Let  $E_i = F[\beta_i]$ , for  $1 \leq i \leq \ell+1$ , and  $E = F[\beta] = \bigoplus_{i=1}^{\ell+1} E_i$ . Set  $\beta' = \beta_{\ell+1}$  and  $E' = E_{\ell+1}$ .

Let  $\widetilde{M}$  be the stabilizer in  $\widetilde{G}$  of  $V = \bigoplus_{j=-m}^m W^{(j)}$ . Then  $\widetilde{M}$  is a  $\sigma$ -stable Levi subgroup of  $\widetilde{G}$ . Let  $\widetilde{P}$  be a  $\sigma$ -stable parabolic subgroup with Levi factor  $\widetilde{M}$ , and  $\widetilde{U}$  the unipotent radical of  $\widetilde{P}$ . We select the Lie algebra of  $\widetilde{U}$  to be elements whose lower triangular block matrices are zero (cf. [1, (7.1.13)]). Then  $P = \widetilde{P} \cap G$ ,  $M^+ = \widetilde{M} \cap G^+$ ,  $M = \widetilde{M} \cap G$  are parabolic subgroups of  $G$  and Levi subgroups of  $G^+$  and  $G$ , respectively. Let  $U = \widetilde{U} \cap G$ . Then  $P = MU$  is a Levi decomposition. There exist isomorphisms

$$M^+ \simeq G^{(0)+} \times \prod_{j=1}^m \widetilde{G}^{(j)}, \quad M \simeq G^{(0)} \times \prod_{j=1}^m \widetilde{G}^{(j)},$$

where  $G^{(0)+}$  is the unitary group of the Hermitian space  $(W^{(0)}, h|_{W^{(0)}})$ ,  $G^{(0)}$  is the identity component of  $G^{(0)+}$ , and  $\widetilde{G}^{(j)} = \text{Aut}_F(W^{(j)})$ , for  $1 \leq j \leq m$ . By (2.3), we have  $\dim_F(W^{(j)}) = N/m$ , for  $j \neq 0$ . Hence there exist isomorphisms

$$\widetilde{G}^{(j)} \simeq GL(N/m, F),$$

for  $1 \leq j \leq m$ .

**Proposition 4.1.** *The subgroups  $H^1(\beta, \Lambda)$ ,  $J^1(\beta, \Lambda)$ ,  $J^0(\beta, \Lambda)$ , and  $J(\beta, \Lambda)$  of  $G$  have Iwahori decompositions with respect to  $(M, P)$ ; letting  $\mathcal{G}$  be any of those groups, we have*

$$\mathcal{G} = (\mathcal{G} \cap U^-)(\mathcal{G} \cap M)(\mathcal{G} \cap U),$$

where  $U^-$  denotes the opposite of  $U$  relative to  $M$ .

*Proof.* Since  $V = \bigoplus_{j=-m}^m W^{(j)}$  is properly subordinated to  $[\Lambda, n, 0, \beta]$  by Definition 2.1 (cf. [14, Definition 6.5]), this follows from [14, Corollary 5.10].

**Proposition 4.2.** (i) *There exists a canonical isomorphism*

$$H^1(\beta, \Lambda) \cap M \simeq H^1(\beta, \Lambda^{(0)}) \times \prod_{j=1}^m \tilde{H}^1(\beta', \Lambda^{(j)}),$$

with corresponding expressions for  $J^1(\beta, \Lambda)$ ,  $J(\beta, \Lambda)$ .

(ii) *Let  $\theta \in \mathcal{C}_-(\Lambda, 0, \beta)$ . Then, under the above isomorphism, we have*

$$\theta|H^1(\beta, \Lambda) \cap M \simeq \theta^{(0)} \otimes \bigotimes_{j=1}^m (\tilde{\theta}^{(j)})^2,$$

where  $\theta^{(0)} \in \mathcal{C}_-(\Lambda^{(0)}, 0, \beta)$ ,  $\tilde{\theta}^{(j)} \in \tilde{\mathcal{C}}(\Lambda^{(j)}, 0, \beta')$ , and  $(\tilde{\theta}^{(j)})^2 \in \tilde{\mathcal{C}}(\Lambda^{(j)}, 0, 2\beta')$ , for  $1 \leq j \leq m$ .

*Proof.* This is a direct consequence of [14, Corollary 5.11].

From Proposition 4.1, we can form the group

$$J_P^1 = H^1(\beta, \Lambda)(J^1(\beta, \Lambda) \cap P).$$

Let  $\theta \in \mathcal{C}_-(\Lambda, 0, \beta)$ , and  $\eta$  be the unique irreducible representation of  $J^1(\beta, \Lambda)$  containing  $\theta$ . Denote by  $\eta_P$  the natural representation of  $J_P^1$  on the subspace of  $(J^1(\beta, \Lambda) \cap U)$ -fixed vectors in the space of  $\eta$ . Then, by [14, Lemma 5.12], we have  $J_P^1 \cap M = J^1(\beta, \Lambda) \cap M$  and

$$\eta_P|J^1(\beta, \Lambda) \cap M \simeq \eta^{(0)} \otimes \bigotimes_{j=1}^m \tilde{\eta}^{(j)},$$

where  $\eta^{(0)}$  is a unique irreducible representation of  $J^1(\beta, \Lambda^{(0)})$  containing  $\theta^{(0)}$ , and  $\tilde{\eta}^{(j)}$  is a unique irreducible representation of  $\tilde{J}^1(\beta', \Lambda^{(j)}) = \tilde{J}^1(2\beta', \Lambda^{(j)})$  containing  $(\tilde{\theta}^{(j)})^2$ .

We define compact open subgroups  $J_P$  and  $J_P^+$  of  $G$  and  $G^+$  by

$$J_P = H^1(\beta, \Lambda)(J(\beta, \Lambda) \cap P), \quad J_P^+ = H^1(\beta, \Lambda)(J^+(\beta, \Lambda) \cap P^+),$$

respectively. Let  $\kappa$  be a  $\beta$ -extension of  $\eta$  to  $J^+(\beta, \Lambda)$ , and  $\kappa_P$  the natural representation of  $J_P^+$  on the space of  $(J^+(\beta, \Lambda) \cap U) = (J^1(\beta, \Lambda) \cap U)$ -fixed vectors in  $\kappa$ . We also denote by  $\kappa_P$  the restriction of  $\kappa_P$  to  $J_P$ . Then from [14, Proposition 6.1],  $\kappa_P|J_P^1 = \eta_P$  and  $\kappa_P$  is irreducible.

**Proposition 4.3.** *The representation  $\kappa_P$  of  $J_P$  satisfies the following conditions:*

- (1)  $\text{Ind}_{J_P}^{J(\beta, \Lambda)} \kappa_P \simeq \kappa|J(\beta, \Lambda)$ ,
- (2) *there exist irreducible representations  $\kappa^{(0)}$  of  $J(\beta, \Lambda^{(0)})$  extending  $\eta^{(0)}$ , and  $\tilde{\kappa}^{(j)}$  of  $\tilde{J}(\beta', \Lambda^{(j)})$  extending  $\tilde{\eta}^{(j)}$ , for  $1 \leq j \leq m$ , (cf. Proposition 6.5) such that*

$$\kappa_P|J_P \cap M \simeq \kappa^{(0)} \otimes \bigotimes_{j=1}^m \tilde{\kappa}^{(j)}.$$

*Proof.* A proof of the proposition can be found below [14, Proposition 5.13].

There exist natural isomorphisms

$$J_P/J_P^1 \simeq J(\beta, \Lambda)/J^1(\beta, \Lambda) \simeq P(\Lambda_{\mathfrak{o}_E})/P_1(\Lambda_{\mathfrak{o}_E}),$$

and, moreover, this quotient is isomorphic to

$$P(\Lambda_{\mathfrak{o}_E}^{(0)})/P_1(\Lambda_{\mathfrak{o}_E}^{(0)}) \times \prod_{j=1}^m \tilde{P}(\Lambda_{\mathfrak{o}_{E'}}^{(j)})/\tilde{P}_1(\Lambda_{\mathfrak{o}_{E'}}^{(j)}).$$

Put  $\overline{G}^{(0)} = P(\Lambda_{\mathfrak{o}_E}^{(0)})/P_1(\Lambda_{\mathfrak{o}_E}^{(0)})$ . Then  $\overline{G}^{(0)}$  is (the group of rational points of) a reductive algebraic group defined over  $k_{F_0}$ , and is not always connected. We denote by  $\overline{G}^\circ$  the identity component of  $\overline{G}^{(0)}$ . Then we have  $\overline{G}^\circ = P^\circ(\Lambda_{\mathfrak{o}_E}^{(0)})/P_1(\Lambda_{\mathfrak{o}_E}^{(0)})$ . From (2.2) and the fact that  $V = \bigoplus_{j=-m}^m W^{(j)}$  is exactly subordinate to  $[\Lambda, n, 0, \beta]$ , the quotient  $\tilde{P}(\Lambda_{\mathfrak{o}_{E'}}^{(j)})/\tilde{P}_1(\Lambda_{\mathfrak{o}_{E'}}^{(j)})$  is isomorphic to  $GL(f, k_{E'})$ . Via these isomorphisms, we identify

$$(4.1) \quad J_P/J_P^1 = J(\beta, \Lambda)/J^1(\beta, \Lambda) = P(\Lambda_{\mathfrak{o}_E})/P_1(\Lambda_{\mathfrak{o}_E}) = \overline{G}^{(0)} \times GL(f, k_{E'})^m.$$

Let  $\tau$  be an irreducible smooth representation of  $J(\beta, \Lambda)$  trivial on  $J^1(\beta, \Lambda)$ . Then  $\tau$  is the inflation to  $J(\beta, \Lambda)$  of the representation

$$\overline{\tau}_0 \otimes \bigotimes_{j=1}^m \overline{\tau}^{(j)},$$

of  $J(\beta, \Lambda)/J^1(\beta, \Lambda)$ , where  $\overline{\tau}_0$  and  $\overline{\tau}^{(j)}$  are representations of  $\overline{G}^{(0)}$  and  $GL(f, k_{E'})$  which are isomorphic to  $J(\beta, \Lambda^{(0)})/J^1(\beta, \Lambda^{(0)})$  and  $\tilde{J}(\beta', \Lambda^{(j)})/\tilde{J}^1(\beta', \Lambda^{(j)})$ , respectively, for  $1 \leq j \leq m$ .

From a  $\beta$ -extension  $\kappa$  as above and  $\tau$ , we define a smooth representation  $\lambda$  of  $J(\beta, \Lambda)$  by

$$\lambda = \kappa \otimes \tau.$$

From (4.1), we can regard  $\tau$  as an irreducible smooth representation of  $J_P$  trivial on  $J_P^1$ .

**Proposition 4.4.** ([14, Lemma 6.1]) *Let  $\lambda_P$  be the natural representation of the group  $J_P = H^1(\beta, \Lambda)(J(\beta, \Lambda \cap P))$  on the space of  $(J^1(\beta, \Lambda) \cap U)$ -fixed vectors in  $\lambda$ . Then*

- (1)  $\lambda_P$  is irreducible and  $\text{Ind}_{J_P}^{J(\beta, \Lambda)} \lambda_P \simeq \lambda$ ,
- (2)  $\lambda_P \simeq \kappa_P \otimes \tau$ ,
- (3) letting  $\lambda^{(0)} = \kappa^{(0)} \otimes \tau_0$  and  $\tilde{\lambda}^{(j)} = \tilde{\kappa}^{(j)} \otimes \tilde{\tau}^{(j)}$ , for  $1 \leq j \leq m$ , we have

$$\lambda_P|_{J_P \cap M} \simeq \lambda^{(0)} \otimes \bigotimes_{j=1}^m \tilde{\lambda}^{(j)},$$

where  $\tau_0$  and  $\tilde{\tau}^{(j)}$  are representations of  $J(\beta, \Lambda^{(0)})$  and  $\tilde{J}(\beta', \Lambda^{(j)})$ , which inflate  $\overline{\tau}_0$  and  $\overline{\tau}^{(j)}$  above, respectively.

**Definition 4.5.** A representation  $\lambda_P = \kappa_P \otimes \tau$  of  $J_P$  as above is called a *simple type* in  $G$ , if  $\tau$  satisfies the following conditions:

- (1)  $\overline{\tau}_0$  is an irreducible representation of  $\overline{G}^{(0)}$  containing an irreducible cuspidal representation of  $\overline{G}^\circ$ ,
- (2)  $\overline{\tau}^{(j)}$  is an irreducible cuspidal representation of  $GL(f, k_{E'})$ , for  $1 \leq j \leq m$ ,
- (3)  $\overline{\tau}^{(1)} \simeq \dots \simeq \overline{\tau}^{(m)}$ .



## 5. SELF-DUAL SIMPLE TYPES

Let  $[\Lambda, n, 0, \beta]$  be a good skew semisimple stratum in  $A$  with splitting  $V = \bigoplus_{i=1}^{\ell+1} V_i$ ,  $\beta = \sum_{i=1}^{\ell+1} \beta_i$ , defined in section 2. Let  $\beta' = \beta_{\ell+1}$ ,  $E_i = F[\beta_i]$ , for  $1 \leq i \leq \ell$ ,  $E' = E_{\ell+1} = F[\beta']$ , and  $E = \bigoplus_{i=1}^{\ell+1} E_i$ . Let  $B$  be the  $A$ -centralizer of  $\beta$ , and  $G_E = B \cap G$ , as before. Then we have

$$G_E = \prod_{j=1}^{\ell+1} G_{E_i},$$

where  $G_{E_i}$  is (the group of rational points of) the restriction of scalars to  $F_0$  of the connected unitary group of  $(V^i, f_i)$ , for  $1 \leq i \leq \ell$ , which is defined in section 2 (cf. [14, p.299]).

Let  $V = \bigoplus_{j=-m}^m W^{(j)}$  be the self-dual decomposition of  $V$  in Definition 2.1. Let  $j > 0$ . We take an (ordered)  $\mathfrak{o}_{E'}$ -basis  $\{v_{j,1}, \dots, v_{j,f}\}$  of the lattice  $\Lambda^{(j)}(0) = \Lambda(0) \cap W^{(j)}$  in  $W^{(j)}$  such that it *splits* the lattice sequence  $\Lambda^{(j)} = \Lambda \cap W^{(j)}$  (see [14, Definition 2.3]), and denote it by  $\mathcal{B}^{(j)}$ . As in [14, 6.2], we take an (ordered)  $E'$ -basis  $\mathcal{B}^{(-j)} = \{v_{-j,1}, \dots, v_{-j,f}\}$  for  $W^{(-j)}$  that satisfies  $f_{\ell+1}(v_{-j,s}, v_{j,t}) = \varpi_{E'} \delta_{s,t}$ , where  $\varpi_{E'}$  is a uniformizer of  $E'$  which satisfies  $\overline{\varpi}_{E'} = (-1)^{e(E'|E'_0)-1} \varpi_{E'}$  and  $\delta_{s,t}$  denotes the Kronecker delta. We also choose a self-dual  $\mathfrak{o}_E$ -basis of the lattice  $\Lambda^{(0)}(0) = \Lambda(0) \cap W^{(0)}$  for  $W^{(0)}$  that splits the lattice sequence  $\Lambda^{(0)} = \Lambda \cap W^{(0)}$ , and denote it by  $\mathcal{B}^{(0)}$ .

Following [14, 6.2] again, we define Weyl group elements of  $G_E$ : For  $j, k \neq 0$ ,  $-m \leq j, k \leq m$ , define  $I_{j,k} \in B' = \text{End}_{E'}(V')$  by  $I_{j,k}(v_{k,s}) = v_{j,s}$  ( $1 \leq s \leq f$ ),  $I_{j,k}(v_{\ell,s}) = 0$  ( $\ell \neq k$ ), where  $V' = \bigoplus_{j=-m, j \neq 0}^m W^{(j)}$ . For  $1 \leq j, k \leq m$ , we let  $s_{j,k}, s_j$ , and  $s_j^\varpi$  be the elements defined in [14, p.333]. Then these elements belong to  $B'$ . Moreover we have  $s_{j,k}, s_j, s_j^\varpi \in G_E^+ = B \cap G^+$ . In particular, the elements  $s_j$  and  $s_j^\varpi$  exchange the blocks  $e^{(j)} A e^{(j)}$  and  $e^{(-j)} A e^{(-j)}$ , where  $e^{(j)}$  denotes the projection  $V \rightarrow W^{(j)}$  with kernel  $\bigoplus_{k \neq j} W^{(k)}$ .

For  $1 \leq j \leq m$ , we define an involution  $\sigma_j$  on  $\tilde{G}^{(j)} = \text{Aut}_F(W^{(j)})$  by using  $s_j$  as follows: Identifying  $\tilde{G}^{(j)} = \{(\overline{g}^{-1}, g) \in \tilde{G}^{(j)} \times \tilde{G}^{(-j)}\}$ , we set

$$\sigma_j(g) = s_j g (s_j)^{-1} \quad (g \in \tilde{G}^{(j)}).$$

**Proposition 5.1.** *Let  $[\Lambda, n, 0, \beta]$  be a good skew semisimple stratum in  $A$ , and  $(J_P, \lambda_P)$  a simple type in  $G$  associated to it with  $\lambda_P = \kappa_P \otimes \tau$ . Let*

$$\kappa_P | J_P \cap M = \kappa^{(0)} \otimes \bigotimes_{j=1}^m \tilde{\kappa}^{(j)} \quad \text{and} \quad \lambda_P | J_P \cap M = \lambda^{(0)} \otimes \bigotimes_{j=1}^m \tilde{\lambda}^{(j)}.$$

*Then we have*

- (1)  $\tilde{J}(\beta', \Lambda^{(1)}) \simeq \dots \simeq \tilde{J}(\beta', \Lambda^{(m)})$  and  $\tilde{J}(\beta', \Lambda^{(j)})$  is  $\sigma_j$ -stable, for  $1 \leq j \leq m$ ,
- (2)  $\tilde{\kappa}^{(1)} \simeq \dots \simeq \tilde{\kappa}^{(m)}$  and  $\tilde{\kappa}^{(j)} \circ \sigma_j \simeq \tilde{\kappa}^{(j)}$ , for  $1 \leq j \leq m$ ,
- (3)  $\kappa^{(0)}$  is a  $\beta$ -extension of  $\eta^{(0)}$ , and  $\tilde{\kappa}^{(j)}$  is a  $2\beta'$ -extension of  $\tilde{\eta}^{(j)}$ , for  $1 \leq j \leq m$ ,
- (4)  $\tilde{\lambda}^{(1)} \simeq \dots \simeq \tilde{\lambda}^{(m)}$ .

*Proof.* By definition 2.1, parts (1), (2), and (3) follow directly from [14, Lemma 6.9, Corollary 6.10, and Proposition 6.3], respectively. For part (4), we have  $\tilde{\lambda}^{(j)} = \tilde{\kappa}^{(j)} \otimes \tilde{\tau}^{(j)}$  by Proposition 4.4. Thus part (2) and Definition 4.5 show part (4). The proof is complete.

**Definition 5.2.** (Selfdual simple type) Let  $(J_P, \lambda_P)$  be a simple type in  $G$  attached to a good skew semisimple stratum  $[\Lambda, n, 0, \beta]$  in  $A$  with  $\lambda_P = \kappa_P \otimes \tau$ . The simple type  $(J_P, \lambda_P)$  in  $G$  is called *self-dual* if the representation  $\tilde{\tau}^{(j)}$  in Proposition 4.4 satisfies  $\tilde{\tau}^{(j)} \circ \sigma_j \simeq \tilde{\tau}^{(j)}$ , for  $1 \leq j \leq m$ .

**Proposition 5.3.** Let  $(J_P, \lambda_P)$  be a simple type in  $G$  attached to a good skew semisimple stratum  $[\Lambda, n, 0, \beta]$  with  $\lambda_P = \kappa_P \otimes \tau$ . Let  $\lambda^{(0)}, \tilde{\lambda}^{(j)}$  be as in Proposition 5.2 for  $\lambda_P|_{J_P \cap M}$ . Then  $\lambda^{(0)}$  and  $\tilde{\lambda}^{(j)}$  are maximal simple types in  $G^{(0)}$  and in  $\tilde{G}^{(j)}$ , for  $1 \leq j \leq m$ , respectively. Moreover, if  $\lambda_P = \kappa_P \otimes \tau$  is self-dual, then  $\tilde{\lambda}^{(j)} \circ \sigma_j \simeq \tilde{\lambda}^{(j)}$ , for  $1 \leq j \leq m$ .

*Proof.* Since  $V = \bigoplus_{j=-m}^m W^{(j)}$  is exactly subordinate to  $[\Lambda, n, 0, \beta]$  in Definition 2.1, it follows from [14, Proposition 6.3 and Definition 6.17] that  $\lambda^{(0)}, \tilde{\lambda}^{(j)}$  are maximal simple types.

From Proposition 5.2, for  $\tilde{\lambda}^{(j)} = \tilde{\kappa}^{(j)} \otimes \tilde{\tau}^{(j)}$ , we have  $\tilde{\kappa}^{(j)} \circ \sigma_j \simeq \tilde{\kappa}^{(j)}$ , for  $1 \leq j \leq m$ . If  $\lambda_P$  is self-dual, by Definition 5.2, we have

$$\begin{aligned} \tilde{\lambda}^{(j)} \circ \sigma_j &\simeq (\tilde{\kappa}^{(j)} \otimes \tilde{\tau}^{(j)}) \circ \sigma_j \\ &\simeq (\tilde{\kappa}^{(j)} \circ \sigma_j) \otimes (\tilde{\tau}^{(j)} \circ \sigma_j) \\ &\simeq \tilde{\kappa}^{(j)} \otimes \tilde{\tau}^{(j)} = \tilde{\lambda}^{(j)}, \end{aligned}$$

for  $1 \leq j \leq m$ . The proof is complete.

## 6. $G$ -COVERS

Let  $[\Lambda, n, 0, \beta]$  be a good skew semisimple stratum in  $A$  with splitting  $V = \bigoplus_{i=1}^{\ell+1} V_i$ ,  $\beta = \sum_{i=1}^{\ell+1} \beta_i$ , defined in section 4, and  $(J_P, \lambda_P)$  a simple type in  $G$  attached to  $[\Lambda, n, 0, \beta]$ . Let  $E_i = F[\beta_i]$ , for  $1 \leq i \leq \ell+1$ ,  $E = \bigoplus_{i=1}^{\ell+1} E_i$ , and  $G_E$  be the  $G$ -centralizer of  $\beta$ . We have  $G_E = \prod_{i=1}^{\ell+1} G_{E_i}$  as in section 5.

As in [14, 6.3], let  $T_{E_i}$  be the maximal split torus of  $G_{E_i}$  which corresponds to the basis  $\mathcal{B}^{(0)} \cap V^i$ , for  $1 \leq i \leq \ell$ , and  $T_{E_{\ell+1}}$  the one of  $G_{E_{\ell+1}}$  corresponding to the basis  $(\bigcup_{j=-m, j \neq 0}^m \mathcal{B}^{(j)}) \cup (\mathcal{B}^{(0)} \cap V^{\ell+1})$  (see (2.1) and (2.2)). Put  $T_E = \prod_{i=1}^{\ell+1} T_{E_i}$ , and let  $N$  be the normalizer of  $T_E$  in  $G_E$ . Put  $N_\Lambda = \{w \in N | w \text{ normalizes } P^\circ(\Lambda_{\mathfrak{o}_E}) \cap M\}$ , as in [14, 6.3]. For the elements  $s_j$ , and  $s_j^\varpi$ , defined in section 5, put  $\zeta_j = \varepsilon s_j s_j^\varpi$ , for  $1 \leq j \leq m$ . We may arrange the order of the basis elements in  $\bigcup_{j=-m}^m \mathcal{B}^{(j)}$  so that the element  $\zeta_j$  of  $T_E$  has a diagonal block form:

$$\zeta_j = \text{Diag}(\underbrace{1_f, \dots, 1_f, \varpi_{E'} 1_f}_{m+1-j}, 1_f, \dots, 1_f, 1_{W^{(0)}}, 1_f, \dots, 1_f, \underbrace{\overline{\varpi}_{E'}^{-1} 1_f, 1_f, \dots, 1_f}_{m+1-j}),$$

where  $1_f$  and  $1_{W^{(0)}}$  denotes the identity matrix in  $\text{End}_{E'}(W^{(j)})$  and  $\text{End}_E(W^{(0)})$  respectively. Denote by  $\mathbf{D}_\Lambda$  the abelian subgroup of  $N_\Lambda$  generated by  $\zeta_j$ , for  $1 \leq j \leq m$ . Then  $\mathbf{D}_\Lambda$  consists of elements

$$\prod_{j=1}^m \zeta_j^{n_{m+1-j}} = \text{Diag}(\varpi_{E'}^{n_1} 1_f, \dots, \varpi_{E'}^{n_m} 1_f, 1_{W^{(0)}}, \overline{\varpi}_{E'}^{-n_m} 1_f, \dots, \overline{\varpi}_{E'}^{-n_1} 1_f),$$

for  $(n_1, \dots, n_m) \in \mathbb{Z}^m$ . Thus there exists an isomorphism  $\mathbb{Z}^m \simeq \mathbf{D}_\Lambda$ .

For the simple type  $\lambda_P = \kappa_P \otimes \tau$  in  $G$ , let  $\rho$  be the irreducible cuspidal component of  $\tau|_{P^0(\Lambda_{\mathfrak{o}_E})}$  (cf. Definition 7.3), and set

$$N_\Lambda(\rho) = \{w \in N_\Lambda \mid {}^w \rho \simeq \rho\}.$$

Then clearly  $\mathbf{D}_\Lambda \subset N_\Lambda(\rho)$ . Denote by  $I_G(\lambda_P)$  the space of  $G$ -intertwiners of  $\lambda_P$ , that is,  $I_G(\lambda_P) = \{g \in G \mid I_g(\lambda_P) \neq (0)\}$ .

**Proposition 6.1.** *Let  $(J_P, \lambda_P)$  be a simple type in  $G$  attached to a good skew semisimple stratum  $[\Lambda, n, 0, \beta]$  in  $A$ . Then we have  $I_G(\lambda_P) \subset J_P N_\Lambda(\rho) J_P$ .*

*Proof.* Suppose that  $g$  intertwines  $\lambda_P = \kappa_P \otimes \tau$ . Then we may assume that  $g \in G_E$ , since  $I_G(\eta_P|J_P^1) = J_P^0 G_E J_P^0$ . In a similar way to the proof of [14, Lemma 5.12], by Clifford Theory, the restriction of  $\lambda_P$  to  $J_P^0$  has the form

$$\lambda_P|J_P^0 = k \sum_p \{(\kappa_P|J_P^0) \otimes \rho\}^p,$$

where  $k$  is the multiplicity and the sum is taken over a set of representatives  $P(\Lambda_{\mathfrak{o}_E})/N_{P(\Lambda_{\mathfrak{o}_E})}(\tau)$ . Since  $g$  intertwines  $\lambda_P|J_P^0$ , there exist  $p_1, p_2 \in P(\Lambda_{\mathfrak{o}_E})$  such that  $p_1 g p_2$  intertwines  $(\kappa_P|J_P^0) \otimes \rho$ . Thus, since  $P(\Lambda_{\mathfrak{o}_E}) \subset J_P$ , we may assume that  $g$  intertwines  $(\kappa_P|J_P^0) \otimes \rho$ . Hence, from [14, Corollary 6.16],  $g \in J_P^0 N_\Lambda(\rho) J_P^0$ , whence  $g \in J_P N_\Lambda(\rho) J_P$ . This completes the proof.

Let  $\pi$  be a smooth representation of  $GL(N/m, F)$ . Let  $\pi^\vee$  be the contragredient representation of  $\pi$ , and define the representation  $\pi^*$  of  $GL(N/m, F)$  by

$$\pi^*(g) = \pi^\vee(\bar{g}) \quad (g \in GL(N/m, F)).$$

A smooth representation  $\pi$  of  $GL(N/m, F)$  is called  $F/F_0$ -selfdual, if it satisfies  $\pi^* \simeq \pi$ , (cf. [11, 12]).

Identifying the Levi subgroup  $M$  with  $G^{(0)} \times GL(N/m, F)^m$ , set

$$J_M = J_P \cap M, \quad \lambda_M = \lambda_P|J_M.$$

**Proposition 6.2.** *Let  $(J_P, \lambda_P)$  be a simple type in  $G$  attached to a good skew semisimple stratum  $[\Lambda, n, 0, \beta]$  in  $A$ . Then there exist irreducible supercuspidal representations  $\pi_{\text{cusp}}$  of  $G^{(0)}$  and  $\tilde{\pi}^{(1)}, \dots, \tilde{\pi}^{(m)}$  of  $GL(N/m, F)$  such that the representations  $\tilde{\pi}^{(1)}, \dots, \tilde{\pi}^{(m)}$  form a single inertial equivalence class, and such that  $(J_M, \lambda_M)$  is a  $[\pi_M, M]_M$ -type in  $M$ , where  $\pi_M = \pi_{\text{cusp}} \otimes \bigotimes_{j=1}^m \tilde{\pi}^{(j)}$ . If  $(J_P, \lambda_P)$  is self-dual, the representations  $\tilde{\pi}^{(1)}, \dots, \tilde{\pi}^{(m)}$  are inertially equivalent to a single irreducible  $F/F_0$ -selfdual supercuspidal representation.*

*Proof.* From Proposition 5.3,  $\lambda_M = \lambda^{(0)} \otimes \bigotimes_{j=1}^m \tilde{\lambda}^{(j)}$ , and  $\lambda^{(0)}$  and the  $\tilde{\lambda}^{(j)}$ 's are all maximal simple types. From [14, Theorem 7.14] and [1, (6.2.3)], there exist an irreducible supercuspidal representation  $\pi_{\text{cusp}}$  of  $G^{(0)}$  containing  $\lambda^{(0)}$  and an irreducible supercuspidal representation  $\tilde{\pi}^{(j)}$  of  $GL(N/m, F)$  containing  $\tilde{\lambda}^{(j)}$ , for  $1 \leq j \leq m$ . Since  $\tilde{\lambda}^{(1)} \simeq \dots \simeq \tilde{\lambda}^{(m)}$ , then, again from [1, (6.2.3)],  $\tilde{\pi}^{(1)}, \dots, \tilde{\pi}^{(m)}$  are mutually inertially equivalent.

Suppose that  $(J_P, \lambda_P)$  is self-dual. Fix  $j$ ,  $1 \leq j \leq m$ . Then, from Proposition 5.3, we have  $\tilde{\lambda}^{(j)} \circ \sigma_j \simeq \tilde{\lambda}^{(j)}$ . Thus  $\tilde{\pi}^{(j)} \circ \sigma_j$  contains  $\tilde{\lambda}^{(j)}$ , and so there exists an unramified character  $\chi$  of  $GL(N/m, F)$  such that  $\tilde{\pi}^{(j)} \circ \sigma_j \simeq \tilde{\pi}^{(j)} \chi$ . Define a

representation  $\pi'$  of  $GL(N/m, F)$  by  $\pi' = \tilde{\pi}^{(j)}\chi^{1/2}$ . Then we have

$$\begin{aligned}\pi' \circ \sigma_j &= (\tilde{\pi}^{(j)}\chi^{1/2}) \circ \sigma_j &= (\tilde{\pi}^{(j)} \circ \sigma_j)(\chi^{1/2} \circ \sigma_j) \\ &\simeq (\tilde{\pi}^{(j)}\chi)\chi^{-1/2} = \pi'.\end{aligned}$$

Employing a theorem of Gelfand and Kazhdan [8, Theorem 2], we have, for  $g \in G$ ,

$$\pi' \circ \sigma_j(g) = \pi'({}^t\bar{g}^{-1}) \simeq (\pi')^\vee(\bar{g}) = (\pi')^*(g).$$

Hence  $(\pi')^* \simeq \pi' \circ \sigma_j \simeq \pi'$ , that is,  $\pi' = \tilde{\pi}^{(j)}\chi$  is inertially equivalent to  $\tilde{\pi}^{(j)}$ , and is  $F/F_0$ -selfdual. This holds for any  $j$ ,  $1 \leq j \leq m$ , whence the proof is completed.

For the parabolic subgroup  $P = MU$ , let  $P^-$  be the opposite parabolic subgroup of  $P$ , and  $U^-$  the unipotent radical of  $P^-$  with  $P^- = MU^-$ . The pair  $(J_M, \lambda_M)$  satisfies the following conditions:

- (1)  $(J_P, \lambda_P)$  is a decomposed pair with respect to  $(M, P)$ , that is,

$$J_P = (J_P \cap U^-)J_M(J_P \cap U),$$

by Proposition 4.1, and  $\lambda_P$  is trivial on  $J_P \cap U$  and  $J_P \cap U^-$ ,

- (2)  $\lambda_M = \lambda_P|_{J_P \cap M}$ .

**Theorem 6.3.** *Let  $(J_P, \lambda_P)$  be a simple type in  $G$  attached to a good skew semisimple stratum  $[\Lambda, n, 0, \beta]$  in  $A$ , and  $\pi_M$  the irreducible supercuspidal representation of  $M$  corresponding to  $(J_M, \lambda_M)$  as in Proposition 6.2. Then  $(J_P, \lambda_P)$  is a  $G$ -cover of  $(J_M, \lambda_M)$ , and so it is an  $[M, \pi_M]_G$ -type in  $G$ .*

*Proof.* By [2, (8.3)], it is necessary to show that there exists an invertible element  $\xi$  of  $\mathcal{H}(G, \lambda_P)$  which is supported on  $J_P z_P J_P$ , where  $z_P$  is an element of the center of  $M$  and is strongly  $(P, J_P)$ -positive (cf. [2, (6.16)]). Define an element  $z_P$  of the center of  $M$  by

$$z_P = \prod_{j=1}^m \zeta_j^{(m-j+1)e(E'|F)} \in D_\Lambda^-.$$

Then, from Proposition 6.3, there exists an element  $\xi' \in \mathcal{H}(M, \lambda_M)$  supported on  $J_M z_P$ , and thus  $\xi = j_P(\xi') \in \mathcal{H}(G, \lambda_P)$  is the desired element. The proof is complete.

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